

Rings Whose Annihilating-Ideal Graphs Have Positive Genus^{*}

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Abstract

Let R be a commutative ring and $\mathbb{A}(R)$ be the set of ideals with non-zero annihilators. The annihilating-ideal graph of R is defined as the graph $\mathbb{AG}(R)$ with the vertex set $\mathbb{A}(R)^* = \mathbb{A} \setminus \{(0)\}$ and two distinct vertices I and J are adjacent if and only if $IJ = (0)$. We investigate commutative rings R whose annihilating-ideal graphs have positive genus $\gamma(\mathbb{AG}(R))$. It is shown that if R is an Artinian ring such that $\gamma(\mathbb{AG}(R)) < \infty$, then R has finitely many ideals or (R, \mathfrak{m}) is a Gorenstein ring with maximal ideal \mathfrak{m} and $\text{v.dim}_{R/\mathfrak{m}} \mathfrak{m}/\mathfrak{m}^2 = 2$. Also, for any two integers $g \geq 0$ and $q > 0$, there are finitely many isomorphism classes of Artinian rings R satisfying the conditions: (i) $\gamma(\mathbb{AG}(R)) = g$ and (ii) $|R/\mathfrak{m}| \leq q$ for every maximal ideal \mathfrak{m} of R . Also, it is shown that if R is a non-domain Noetherian local ring such that $\gamma(\mathbb{AG}(R)) < \infty$, then either R is a Gorenstein ring or R is an Artinian ring with finitely many ideals.

Key Words: Commutative ring; annihilating-ideal graph; genus of a graph.

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1 Introduction

The study of algebraic structures, using the properties of graphs, became an exciting research topic in the last twenty years, leading to many fascinating results and

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questions. There are many papers on assigning a graph to a ring, for instance see, [3], [4], [5], [6] and [7]. Throughout this paper, R denotes a commutative ring with $1 \neq 0$. Let X be either an element or a subset of R . The annihilator of X is the ideal $\text{Ann}(X) = \{a \in R : aX = 0\}$. We note that an element $a \in R$ is called a zero-divisor in R if $\text{Ann}(a) \neq \{0\}$ (i.e., $ab = 0$ for some nonzero element $b \in R$). For any subset Y of R , we denote $|Y|$ for the cardinality of Y and let $Y^* = Y \setminus \{0\}$. Also, we denote the finite field of order q by \mathbb{F}_q . Let V be a vector space over the field \mathbb{F} . We use the notation $\text{v.dim}_{\mathbb{F}}(V)$ to denote the dimension of V over the field \mathbb{F} .

Let S_k denote the sphere with k handles, where k is a non-negative integer, that is, S_k is an oriented surface of genus k . The *genus* of a graph G , denoted $\gamma(G)$, is the minimal integer n such that the graph can be embedded in S_n . For details on the notion of embedding a graph in a surface, see, e.g., [16, Chapter 6]. Intuitively, G is embedded in a surface if it can be drawn in the surface so that its edges intersect only at their common vertices. A genus 0 graph is called a *planar graph* and a genus 1 graph is called a *toroidal graph*. An infinite graph G has infinite genus ($\gamma(G) = \infty$), if, for every natural number n , there exists a finite subgraph G_n of G such that $\gamma(G_n) = n$. We note here that if H is a subgraph of a graph G , then $\gamma(H) \leq \gamma(G)$. Let K_n denote the complete graph on n vertices; that is, K_n has vertex set V with $|V| = n$ and $a - b$ is an edge for every distinct $a, b \in V$. Let $K_{m,n}$ denote the complete bipartite graph; that is, $K_{m,n}$ has vertex set V consisting of the disjoint union of two subsets, V_1 and V_2 , such that $|V_1| = m$ and $|V_2| = n$, and $a - b$ is an edge if and only if $a \in V_1$ and $b \in V_2$. We may sometimes write $K_{|V_1|, |V_2|}$ to denote the complete bipartite graph with vertex sets V_1 and V_2 . Note that $K_{m,n} = K_{n,m}$. It is well known that

$$\gamma(K_n) = \lceil \frac{(n-3)(n-4)}{12} \rceil \quad \text{for all } n \geq 3.$$

$$\gamma(K_{m,n}) = \lceil \frac{(n-2)(m-2)}{4} \rceil \quad \text{for all } n \geq 2 \text{ and } m \geq 2.$$

(see [12]; [10], respectively). For a graph G , the degree of a vertex v of G is the number of edges of G incident with v .

Let R be a ring. We call an ideal I of R , an *annihilating-ideal* if there exists a non-zero ideal J of R such that $IJ = (0)$. We denote $Z(R)$ the sets of zero-divisors, $\mathbb{A}(R)$ for the set of all annihilating-ideals of R , $J(R)$ for the Jacobson radical of R , and for an ideal J of R , we denote $\mathbb{I}(J)$ for the set $\{I : I \text{ be an ideal of } R \text{ and } I \subseteq J\}$. Also, by the *annihilating-ideal graph* $\mathbb{AG}(R)$ of R we mean the graph with vertices $\mathbb{AG}(R)^* = \mathbb{A}(R) \setminus \{(0)\}$ such that there is an (undirect) edge between vertices I and J if and only if $I \neq J$ and $IJ = (0)$. Thus $\mathbb{AG}(R)$ is an empty graph if and only if R is an integral domain. Recently, the idea of annihilating-ideal graph of commutative rings

were introduced by Behboodi and Rakeei in [6], [7] and by Behboodi et al., in [1]. The *zero-divisor graph* of R , $\Gamma(R)$, is an undirect graph with the vertex set $Z(R) \setminus \{0\}$ and two distinct vertices x, y are adjacent if and only if $xy = (0)$. There are some papers which investigate genus of a zero-divisor graph, for instance see [14], [15] and [13]. In particular in [15, Theorem 2], it was shown that for any positive integer g , there are finitely many finite commutative rings whose zero-divisor graph has genus g .

In the present paper we investigate rings whose annihilating-ideal graphs have positive genus. In Theorem 2.7, it is shown that if R is an Artinian ring and $\gamma(\mathbb{A}\mathbb{G}(R)) < \infty$, then R has finitely many ideals or (R, \mathfrak{m}) is Gornestein ring with $\dim_{R/\mathfrak{m}} \mathfrak{m}/\mathfrak{m}^2 = 2$ (a Noetherian local ring (R, \mathfrak{m}) is called *Gorenstein* if $\text{v.dim}_{R/\mathfrak{m}}(\text{Ann}(\mathfrak{m})) = 1$). Also, it is shown that, for two integers $g \geq 0$ and $q > 0$, there are finitely many Artinian rings R such that (i) $\gamma(\mathbb{A}\mathbb{G}(R)) = g$ and (ii) $|R/\mathfrak{m}| \leq q$ for every maximal ideal \mathfrak{m} of R (see Theorem 2.10 and Corollary 2.11). Also, the genus of annihilating-ideal graphs of Noetherian rings are studied. It is shown that if R is a Noetherian local ring and $\gamma(\mathbb{A}\mathbb{G}(R)) < \infty$, then either R is a domain or all non-trivial ideals of R are vertices of $\mathbb{A}\mathbb{G}(R)$ (see Proposition 3.4). Also if R is a Noetherian ring such that all non-trivial ideals of R are vertices of $\mathbb{A}\mathbb{G}(R)$ and $\gamma(\mathbb{A}\mathbb{G}(R)) < \infty$, then either R is a Gorenstein ring or R is an Artinian ring with finitely many ideals (see Theorem 3.5). Finally, it is shown that if R is a non-domain Noetherian local ring such that $\gamma(\mathbb{A}\mathbb{G}(R)) < \infty$, then either R is a Gorenstein ring or R is an Artinian ring with finitely many ideals (see Corollary 3.6).

2 The Genus of the Annihilating-Ideal Graphs of Artinian Rings

The following useful lemma and remark will be used frequently in this paper.

Lemma 2.1 [6, Proposition 1.3] *Let R be an Artinian ring. Then every nonzero proper ideal I is a vertex of $\mathbb{A}\mathbb{G}(R)$.*

Remark 2.2 It is well known that if V is a vector space over an infinite field \mathbb{F} , then V can not be the union of finitely many proper subspaces; see for example [8, p.283].

A local Artinian principal ring is called a *special principal ring* and has an extremely simple ideal structure: there are only finitely many ideals, each of which is a power of the maximal ideal.

Lemma 2.3 *Let (R, \mathfrak{m}) be a local ring with $\mathfrak{m}^t = (0)$. If for a positive integer n , $\text{v.dim}_{R/\mathfrak{m}}(\mathfrak{m}^n/\mathfrak{m}^{n+1}) = 1$ and \mathfrak{m}^n is a finitely generated R -module, Then $\mathbb{I}(\mathfrak{m}^n) = \{\mathfrak{m}^i : t \leq i \leq n\}$. Moreover If $n = 1$, then R is a special principal ring.*

Proof. Since $\text{v.dim}_{R/\mathfrak{m}}(\mathfrak{m}^n/\mathfrak{m}^{n+1}) = 1$, by using Nakayama's lemma, $\mathfrak{m}^n = Rx$ for some $x \in R$. Now, let I be a nonzero ideal of R such that $I \subseteq \mathfrak{m}^n$. Since $\mathfrak{m}^t = (0)$, there exists a natural number $i \geq n$ such that $I \subseteq \mathfrak{m}^i$ and $I \not\subseteq \mathfrak{m}^{i+1}$. Let $a \in I \setminus \mathfrak{m}^{i+1}$. We have $a = bx^i$, for some $b \in R$. If $b \in \mathfrak{m}$, then $a \in \mathfrak{m}^{i+1}$, a contradiction. Thus b is unit. Hence $x^i \in I$. This implies that $I = (x^i)$, as desired. Thus $\mathbb{I}(\mathfrak{m}^n) = \{\mathfrak{m}^i : t \leq i \leq n\}$. Clearly, if $n = 1$, then R is a special principal ring. \square

Lemma 2.4 *Let (R, \mathfrak{m}) be an Artinian local ring with $|R/\mathfrak{m}| = \infty$, $\mathfrak{m}^3 = (0)$ and $\text{v.dim}_{R/\mathfrak{m}}\text{Ann}(\mathfrak{m}) = 1$. If $\gamma(\mathbb{A}\mathbb{G}(R)) < \infty$, then $\text{v.dim}_{R/\mathfrak{m}}\mathfrak{m}/\mathfrak{m}^2 \leq 2$*

Proof. First we assume that $\text{v.dim}_{R/\mathfrak{m}}(\mathfrak{m}/\mathfrak{m}^2) = 3$. Then there exist infinitely many subspaces with dimension one. Let Rx/\mathfrak{m}^2 and Ry/\mathfrak{m}^2 be two distinct subspaces of dimension one of $\mathfrak{m}/\mathfrak{m}^2$. Since $Rx \cong R/\text{Ann}(x)$ and Rx has only one nonzero proper R -submodule, $\mathfrak{m}/\text{Ann}(x)$ is the only nonzero proper ideal of $R/\text{Ann}(x)$. Therefore, $\text{v.dim}_{R/\mathfrak{m}}\text{Ann}(x) = 2$. Similarly, $\text{v.dim}_{R/\mathfrak{m}}\text{Ann}(y) = 2$. Therefore, $|\mathbb{I}(\text{Ann}(x))| = |\mathbb{I}(\text{Ann}(y))| = \infty$. If $\text{Ann}(x) = \text{Ann}(y)$, then since $(Rx)\text{Ann}(x) = (0)$, $(Ry)\text{Ann}(x) = (0)$ and $\mathfrak{m}^2\text{Ann}(x) = (0)$, we conclude that $K_{|\mathbb{I}(\text{Ann}(x))|, 3}$ is a subgraph of $\mathbb{A}\mathbb{G}(R)$. Thus by the formula for genus of complete bipartite graph, $\gamma(\mathbb{A}\mathbb{G}(R)) = \infty$, a contradiction. Thus we may assume that $\text{Ann}(x) \neq \text{Ann}(y)$. If $\text{Ann}(x) \cap \text{Ann}(y) = (0)$, then $(\text{Ann}(x))(\text{Ann}(y)) = (0)$ and so $K_{|\mathbb{I}(\text{Ann}(x))|}$ is a subgraph of $\mathbb{A}\mathbb{G}(R)$, and hence, by the formula for complete graphs $\gamma(\mathbb{A}\mathbb{G}(R)) = \infty$, a contradiction. Therefore, $\text{Ann}(x) \cap \text{Ann}(y) \neq (0)$ and so $\text{v.dim}_{R/\mathfrak{m}}(\text{Ann}(x) \cap \text{Ann}(y))/\mathfrak{m}^2 = 1$.

Suppose that $(Rx)(Ry) \neq (0)$. Since $\text{v.dim}_{R/\mathfrak{m}}(\text{Ann}(x) \cap \text{Ann}(y))/\mathfrak{m}^2 = 1$, there exists ideal K such that $Rx \text{ --- } K \text{ --- } Ry$. Let I_1 be an ideal such that $I_1 \in \mathbb{I}(\text{Ann}(x)) \setminus \{K, Ry\}$ and $\text{v.dim}_{R/\mathfrak{m}}I_1/\mathfrak{m}^2 = 1$. Let J_1 be an ideal such that $J_1 \in \mathbb{I}(\text{Ann}(x)) \setminus \{Rx, Ry, K, I_1, \text{Ann}(x) \cap \text{Ann}(y)\}$ such that $\text{v.dim}_{R/\mathfrak{m}}J_1/\mathfrak{m}^2 = 1$. Let $K_1 = \text{Ann}(I_1) \cap \text{Ann}(J_1)$. Therefore, $Rx \text{ --- } I_1 \text{ --- } K_1 \text{ --- } J_1 \text{ --- } Ry$ is a path between Rx and Ry . Now, let $I_n \in \mathbb{I}(\text{Ann}(x)) \setminus \{I_i, J_i, K_i, Rx, Ry, \text{Ann}(x) \cap \text{Ann}(y), \text{Ann}(x) \cap \text{Ann}(K_i), i = 1, 2, \dots, n-1\}$ such that $\text{v.dim}_{R/\mathfrak{m}}I_n/\mathfrak{m}^2 = 1$ and $J_n \in \mathbb{I}(\text{Ann}(x)) \setminus \{I_i, J_{i-1}, K_{i-1}, I_n, Rx, Ry, \text{Ann}(x) \cap \text{Ann}(y), \text{Ann}(x) \cap \text{Ann}(K_{i-1}), i = 1, 2, \dots, n\}$ such that $\text{v.dim}_{R/\mathfrak{m}}J_n/\mathfrak{m}^2 = 1$. Suppose that $K_n = \text{Ann}(I_n) \cap \text{Ann}(J_n)$, then $Rx \text{ --- } I_n \text{ --- } K_n \text{ --- } J_n \text{ --- } Ry$ is a path between Rx and Ry . Therefore, there exists infinitely many path between Rx and Ry . Thus either $(Rx)(Ry) = (0)$ or there exist infinitely many path between Rx and Ry .

Since Rx/\mathfrak{m}^2 and Ry/\mathfrak{m}^2 are two arbitrary distinct one dimensional R/\mathfrak{m} -subspaces of

$\text{Ann}(x)$ and there are infinitely many subspaces of dimension one of $\text{Ann}(x)$, one can easily see that $\gamma(\mathbb{A}\mathbb{G}(R)) = \infty$, a contradiction.

Now, we assume that $\text{v.dim}_{R/\mathfrak{m}}(\mathfrak{m}/\mathfrak{m}^2) > 3$. Suppose that $\{x_1 + \mathfrak{m}^2, x_2 + \mathfrak{m}^2, x_3 + \mathfrak{m}^2, \dots, x_n + \mathfrak{m}^2\}$ is a basis for $\mathfrak{m}/\mathfrak{m}^2$ over R/\mathfrak{m} . Since $Rx_1 \cong R/\text{Ann}(x_1)$ and Rx_1 has one R -submodule, $\text{v.dim}_{R/\mathfrak{m}}(\text{Ann}(x_1)/\mathfrak{m}^2) = n-1$. Similarly $\text{v.dim}_{R/\mathfrak{m}}(\text{Ann}(x_2)/\mathfrak{m}^2) = n-1$. Therefore, $\text{v.dim}_{R/\mathfrak{m}}(\text{Ann}(x_2) \cap \text{Ann}(x_1))/\mathfrak{m}^2 = n-2$. Since $(Rx_1)(\text{Ann}(x_2) \cap \text{Ann}(x_1)) = (0)$, $(Rx_2)(\text{Ann}(x_2) \cap \text{Ann}(x_1)) = (0)$ and $\mathfrak{m}^2(\text{Ann}(x_2) \cap \text{Ann}(x_1)) = (0)$, we conclude that $K_{|\mathbb{I}((\text{Ann}(x_2) \cap \text{Ann}(x_1)))/\mathfrak{m}^2|, 3}$ is a subgraph of $\gamma(\mathbb{A}\mathbb{G}(R))$. Thus by the formula for genus of complete bipartite graphs $\gamma(\mathbb{A}\mathbb{G}(R)) = \infty$, a contradiction. \square

Theorem 2.5 *Let (R, \mathfrak{m}) be an Artinian local ring. If $\gamma(\mathbb{A}\mathbb{G}(R)) < \infty$, then R has finitely many ideals or R is a Gorenstein ring with $\dim_{R/\mathfrak{m}} \mathfrak{m}/\mathfrak{m}^2 = 2$.*

Proof. If R is a field, then R has finitely many ideals. Thus we can assume that R is not a field. If $|R/\mathfrak{m}| < \infty$, then one can easily see that R is a finite ring and so R has finitely many ideals. Thus we can assume that $|R/\mathfrak{m}| = \infty$ and $\gamma(\mathbb{A}\mathbb{G}(R)) = g$ for an integer $g \geq 0$. The proof now proceeds by cases:

Case 1: $\text{v.dim}_{R/\mathfrak{m}}(\text{Ann}(\mathfrak{m})) \geq 2$. Then $|\mathbb{I}(\text{Ann}(\mathfrak{m}))| = |\mathbb{I}(\mathfrak{m})| = \infty$ and since $\mathfrak{m}\text{Ann}(\mathfrak{m}) = (0)$, $K_{|\mathbb{I}(\mathfrak{m})|-4, 3}$ is a subgraph of $\mathbb{A}\mathbb{G}(R)$. Hence, by the formula for genus of complete bipartite graphs $\lceil (|\mathbb{I}(\mathfrak{m})| - 6)/4 \rceil \leq g$ and so $|\mathbb{I}(\mathfrak{m})| \leq 4g + 6$, a contradiction.

Case 2: $\text{v.dim}_{R/\mathfrak{m}}(\text{Ann}(\mathfrak{m})) = 1$. Since R is an Artinian ring, there exists positive integer t such that $\mathfrak{m}^{t+1} = (0)$ and $\mathfrak{m}^t \neq (0)$.

Subcase 1: $t = 1$, i.e., $\mathfrak{m}^2 = (0)$. Then $K_{|\mathbb{I}(\mathfrak{m})|-1}$ is a subgraph of $\mathbb{A}\mathbb{G}(R)$ and so by the formula for genus of complete graphs, $\lceil (|\mathbb{I}(\mathfrak{m})| - 6)/12 \rceil \leq g$. Hence $|\mathbb{I}(\mathfrak{m})| \leq 12g + 6$, i.e., R has at most $12g + 7$ ideals.

Subcase 2: $t = 2$, i.e., $\mathfrak{m}^2 \neq (0)$ and $\mathfrak{m}^3 = (0)$. Then Lemma 2.4 implies that $\text{v.dim}_{R/\mathfrak{m}}(\mathfrak{m}/\mathfrak{m}^2) \leq 2$. If $\text{v.dim}_{R/\mathfrak{m}}(\mathfrak{m}/\mathfrak{m}^2) = 1$, then Lemma 2.3 implies that $|\mathbb{I}(\mathfrak{m})| = 3$, i.e., R has four ideals.

Subcase 3: $t \geq 3$. Since $\mathfrak{m}^3\mathfrak{m}^t = \mathfrak{m}^3\mathfrak{m}^{t-1} = \mathfrak{m}^3\mathfrak{m}^{t-2} = (0)$, $K_{|\mathbb{I}(\mathfrak{m}^3)|, 3}$ is a subgraph of $\mathbb{A}\mathbb{G}(R)$. So by the formula for genus of complete bipartite graphs, $\lceil (|\mathbb{I}(\mathfrak{m}^3)| - 6)/12 \rceil \leq g$. Hence, $|\mathbb{I}(\mathfrak{m}^3)| \leq 12g + 6$. If $\text{v.dim}_{R/\mathfrak{m}}(\mathfrak{m}^{t-1}/\mathfrak{m}^t) \geq 2$, then Remark 2.2 implies that $|\mathbb{I}(\mathfrak{m}^{t-1})| = \infty$. Since $\mathfrak{m}^{t-1}\mathfrak{m}^{t-1} = (0)$ and $t \geq 3$, $K_{|\mathbb{I}(\mathfrak{m}^{t-1})|-1}$ is a subgraph of $\mathbb{A}\mathbb{G}(R)$. Therefore, $\gamma(\mathbb{A}\mathbb{G}(R)) = \infty$, a contradiction. Thus $\text{v.dim}_{R/\mathfrak{m}}(\mathfrak{m}^{t-1}/\mathfrak{m}^t) = 1$. Hence, by Lemma 2.3, there exists $x \in \mathfrak{m}^{t-1}$ such that $\mathfrak{m}^{t-1} = Rx$.

Suppose that $\text{v.dim}_{R/\mathfrak{m}} \mathfrak{m}/\mathfrak{m}^2 = n \geq 3$. Since $Rx \cong R/\text{Ann}(x)$ and Rx has only one nonzero proper R -submodule, $\mathfrak{m}/\text{Ann}(x)$ is the only nonzero proper ideal of $R/\text{Ann}(x)$. If $\mathfrak{m}^2 \not\subseteq \text{Ann}(y)$, then $\text{Ann}(x) + \mathfrak{m}^2 = \mathfrak{m}$, and by Nakayama's lemma, $\text{Ann}(x) = \mathfrak{m}$, a contradiction. Thus $\mathfrak{m}^2 \subseteq \text{Ann}(x)$ and since $\mathfrak{m}/\text{Ann}(x)$ is the only nonzero proper ideal

of $R/\text{Ann}(x)$, $\text{v.dim}_{R/\mathfrak{m}} \text{Ann}(x)/\mathfrak{m}^2 = n - 1$. Let $\{y_1 + \mathfrak{m}^2, y_2 + \mathfrak{m}^2, \dots, y_{n-1} + \mathfrak{m}^2, y + \mathfrak{m}^2\}$ be a basis of $\mathfrak{m}/\mathfrak{m}^2$ such that $\{y_1 + \mathfrak{m}^2, y_2 + \mathfrak{m}^2, \dots, y_{n-1} + \mathfrak{m}^2\}$ be a basis of $\text{Ann}(x)/\mathfrak{m}^2$. Since \mathfrak{m}^t is the only minimal ideal of R , $\mathfrak{m}^t \subseteq Ry$ and $|\mathbb{I}(Ry)| \geq 3$.

Suppose that $|\mathbb{I}(Ry)| = 3$. Since $Ry \cong R/\text{Ann}(y)$ and Ry has only one nonzero proper R -submodule, $\mathfrak{m}/\text{Ann}(y)$ is the only nonzero proper ideal of $R/\text{Ann}(y)$. If $\mathfrak{m}^2 \not\subseteq \text{Ann}(y)$, then $\text{Ann}(y) + \mathfrak{m}^2 = \mathfrak{m}$, and thus by Nakayama's lemma, $\text{Ann}(y) = \mathfrak{m}$, a contradiction. Therefore, $\mathfrak{m}^2 \subseteq \text{Ann}(y)$. So $\mathfrak{m}^{t-1} \subseteq \text{Ann}(y)$. Hence, $\mathfrak{m}^{t-1}\mathfrak{m} = (0)$ ($\mathfrak{m}^{t-1}(Ry) = (0)$ and $\mathfrak{m}^{t-1}(Ry_1 + \dots + Ry_{n-1}) = (0)$), a contradiction. Therefore, $|\mathbb{I}(Ry)| \geq 4$.

Assume $|\mathbb{I}(Ry)| < \infty$. Since $Ry \cong R/\text{Ann}(y)$, $\text{v.dim}_{R/\mathfrak{m}}(\text{Ann}(y) + \mathfrak{m}^2)/\mathfrak{m}^2 = n - 1$. Thus $|\mathbb{I}(\text{Ann}(y))| = \infty$ and $K_{|\mathbb{I}(\text{Ann}(y))|, |\mathbb{I}(Ry)|}$ is a subgraph of $\mathbb{AG}(R)$. So by the formula for genus of complete bipartite graphs, $\gamma(\mathbb{AG}(R)) = \infty$.

So we now assume that $|\mathbb{I}(Ry)| = \infty$. Suppose that $\text{v.dim}_{R/\mathfrak{m}} \mathfrak{m}^2/\mathfrak{m}^3 \geq 2$. If $|\mathbb{I}(\text{Ann}(y))| \geq 4$, then $K_{|\mathbb{I}(Ry)|, 3}$ is a subgraph of $\mathbb{AG}(R)$ and so by the formula for genus of complete bipartite graphs, $\gamma(\mathbb{AG}(R)) = \infty$, a contradiction. We may assume that $|\mathbb{I}(\text{Ann}(y))| = 3$. $\text{Ann}(y) \not\subseteq \mathfrak{m}^3$ (since $\mathfrak{m}^2y \cong \mathfrak{m}^2/(\text{Ann}(y) \cap \mathfrak{m}^2)$, there exist finitely many ideals between $\text{Ann}(y)$ and \mathfrak{m}^2). Therefore, there exists $z \in \text{Ann}(y) \setminus \mathfrak{m}^3$. Since $|\mathbb{I}(\text{Ann}(y))| = 3$, $Rz = \text{Ann}(y)$. Since $Rz \cong R/\text{Ann}(z)$ and Rz has only one nonzero proper R -submodule, $\mathfrak{m}/\text{Ann}(z)$ is the only nonzero proper ideal of $R/\text{Ann}(z)$. If $\mathfrak{m}^2 \not\subseteq \text{Ann}(z)$, then $\text{Ann}(z) + \mathfrak{m}^2 = \mathfrak{m}$, and thus by Nakayama's lemma, $\text{Ann}(z) = \mathfrak{m}$, a contradiction. Therefore, $\mathfrak{m}^2 \subseteq \text{Ann}(z) = \text{Ann}(\text{Ann}(y))$. Since $\mathfrak{m}^2\text{Ann}(y) = (0)$, $\mathfrak{m}^2\mathfrak{m}^{t-1} = (0)$ and $\mathfrak{m}^t\mathfrak{m}^2 = (0)$, $K_{|\mathbb{I}(\mathfrak{m}^2)|, 3}$ is a subgraph of $\mathbb{AG}(R)$. So by the formula for genus of complete bipartite graphs, $\gamma(\mathbb{AG}(R)) = \infty$, a contradiction. Therefore, $\text{v.dim}_{R/\mathfrak{m}} \mathfrak{m}^2/\mathfrak{m}^3 = 1$ and by Lemma 2.3, $|\mathbb{I}(\mathfrak{m}^2)| < \infty$. Since $my \cong \mathfrak{m}/\text{Ann}(y)$ and $|\mathbb{I}(my)| < \infty$, $\text{v.dim}_{R/\mathfrak{m}}(\text{Ann}(y) + \mathfrak{m}^2)/\mathfrak{m}^2 = n - 1$. So $|\mathbb{I}(\text{Ann}(y))| = \infty$. Hence, $K_{|\mathbb{I}(\text{Ann}(y))|, |\mathbb{I}(Ry)|}$ is a subgraph of $\mathbb{AG}(R)$ and by the formula for genus of complete bipartite graphs, $\gamma(\mathbb{AG}(R)) = \infty$, a contradiction. Thus $\text{v.dim}_{R/\mathfrak{m}} \mathfrak{m}/\mathfrak{m}^2 \leq 2$.

Also, if $\text{v.dim}_{R/\mathfrak{m}} \mathfrak{m}/\mathfrak{m}^2 = 1$, then by Lemma 2.3, R has finitely many ideals. \square

Lemma 2.6 *Let $R = \prod_{i \in I} R_i$ be a product of nonzero rings R_i ($i \in I$). If $\gamma(\mathbb{AG}(R)) < \infty$, then I is finite and for each $i \in I$, $\gamma(\mathbb{AG}(R_i)) < \infty$. Consequently, if R is Artinian, then $\gamma(\mathbb{AG}(R)) < \infty$ if and only if $\gamma(\mathbb{AG}(R_i)) < \infty$ for every $i \in I$.*

Proof. Suppose, contrary to our claim, that $|I| = \infty$. Let $I_1 = (R_{i_1} \times R_{i_2} \times 0 \times \dots \times 0)$ and $I_2 = (0 \times 0 \times R_{i_3} \times R_{i_4} \times \dots)$. Since $I_1 I_2 = (0)$, $K_{|\mathbb{I}(I_1)|, |\mathbb{I}(I_2)|}$ is a subgraph of $\mathbb{AG}(R)$. Since $|\mathbb{I}(I_1)| \geq 4$ and $|\mathbb{I}(I_2)| = \infty$, by the formula for genus of complete bipartite graphs, $\gamma(\mathbb{AG}(R)) = \infty$, a contradiction. Therefore, $|I| < \infty$ and since $\mathbb{AG}(R_i)$ is a subgraph of $\mathbb{AG}(R)$ for every $i \in I$ and $\gamma(\mathbb{AG}(R)) < \infty$, we conclude that $\gamma(\mathbb{AG}(R_i)) < \infty$ for every $i \in I$.

Now, suppose that R is an Artinian ring. It is well known that $R \cong R_1 \times \dots \times R_n$ for some positive integer n , where every R_i ($i = 1, 2, \dots, n$) is an Artinian local ring. If $n = 1$, then the proof is immediate. Now, we may assume that $n \geq 2$. If every R_i has finitely many ideals, then R has finitely many ideals and therefore $\gamma(\mathbb{A}\mathbb{G}(R)) < \infty$. Without loss of generality, we can assume that R_1 has infinitely many ideals. Let $I_1 = (0 \times R_2 \times 0 \times \dots \times 0)$, $I_2 = (\text{Ann}(\mathfrak{m}_1) \times 0 \times \dots \times 0)$ and $I_3 = (\text{Ann}(\mathfrak{m}_1) \times R_2 \times 0 \times \dots \times 0)$. Then for every ideal J of R_1 , we have $I_i(J \times 0 \times \dots \times 0) = (0)$. Therefore, $K_{|\mathbb{I}(\mathfrak{m}_1)|-1,3}$ is a subgraph of $\mathbb{A}\mathbb{G}(R)$. Since $|\mathbb{I}(\mathfrak{m}_1)| = \infty$, by the formula for genus of complete bipartite graphs, $\gamma(\mathbb{A}\mathbb{G}(R)) = \infty$, a contradiction. Therefore, every R_i has finitely many ideals and $\gamma(\mathbb{A}\mathbb{G}(R)) < \infty$. \square

Theorem 2.7 *Let R be an Artinian ring. If $\gamma(\mathbb{A}\mathbb{G}(R)) < \infty$, then R has finitely many ideals or (R, \mathfrak{m}) is a Gorenstein ring with $\text{v.dim}_{R/\mathfrak{m}} \mathfrak{m}/\mathfrak{m}^2 = 2$.*

Proof. Let R be an Artinian ring. It is well known that $R \cong R_1 \times \dots \times R_n$ for some positive integer n , where every R_i ($i = 1, 2, \dots, n$) is an Artinian local ring and addition and multiplication in the product are defined component wise. If R is a local ring, then by Theorem 2.5, R has finitely many ideals or $\text{v.dim}_{R/\mathfrak{m}} \mathfrak{m}/\mathfrak{m}^2 = 2$. If $n \geq 2$, then as in the proof of Lemma 2.6, every R_i has finitely many ideals and therefore R has finitely many ideals. \square

Lemma 2.8 *Let (R, \mathfrak{m}) be a Gorenstein ring, k be a positive integer and q be a prime power. If $\mathfrak{m}^2 \neq (0)$, $\mathfrak{m}^3 = (0)$, $|R/\mathfrak{m}| = q$ and $\text{v.dim}_{R/\mathfrak{m}}(\mathfrak{m}/\mathfrak{m}^2) = k > 6$, then $\mathbb{A}\mathbb{G}(R)$ contains a copy of $K_{k-6,3}$.*

Proof. Since $\text{v.dim}_{R/\mathfrak{m}}(\mathfrak{m}/\mathfrak{m}^2) = k$ and $|R| = |R/\mathfrak{m}| |\mathfrak{m}/\mathfrak{m}^2| |\mathfrak{m}^2|$, we conclude that $|R| = q^{k+2}$. Let $\{x_1 + \mathfrak{m}^2, \dots, x_k + \mathfrak{m}^2\}$ be a basis for $\mathfrak{m}/\mathfrak{m}^2$ over R/\mathfrak{m} . We have $R/\text{Ann}(x_1) \cong Rx_1$ thus $|R| = |Rx_1| |\text{Ann}(x_1)|$. Since $\mathfrak{m}^2 \subseteq Rx_1$, $|Rx_1| = |\mathfrak{m}^2| |q| = q^2$ and $|R| = q^2 |\text{Ann}(x_1)|$. Therefore, $|\text{Ann}(x_1)| = q^k$ and $\text{v.dim}_{R/\mathfrak{m}}(\text{Ann}(x_1)/\mathfrak{m}^2) = k - 2$. Similarly $\text{v.dim}_{R/\mathfrak{m}}(\text{Ann}(x_2)/\mathfrak{m}^2) = k - 2$. Thus $\text{v.dim}_{R/\mathfrak{m}}((\text{Ann}(x_1)/\mathfrak{m}^2) \cap (\text{Ann}(x_2)/\mathfrak{m}^2)) \geq k - 4$. Let $\{x_{i_1} + \mathfrak{m}^2, \dots, x_{i_n} + \mathfrak{m}^2\}$ be a basis of $(\text{Ann}(x_1)/\mathfrak{m}^2) \cap (\text{Ann}(x_2)/\mathfrak{m}^2)$. Suppose that $V_1 = \{Rx_1, Rx_2, \mathfrak{m}^2\}$ and $V_2 = \{Rx_{i_1}, \dots, Rx_{i_n}\}$. Since $K_{|V_1|,|V_2|}$ is a subgraph of $\mathbb{A}\mathbb{G}(R)$, $K_{k-6,3}$ is a subgraph of $\mathbb{A}\mathbb{G}(R)$. \square

Lemma 2.9 *Let (R, \mathfrak{m}) be an Artinian local ring with $|R/\mathfrak{m}| = q$ such that R has finitely many ideals. Then $|\mathfrak{m}^i| \leq q^{\mathbb{I}(\mathfrak{m}^i)}$ for all i and $|R| \leq q^{\mathbb{I}(R)}$*

Proof. If R is a field, then the proof is straightforward. Thus we can assume that R is not a field. Since R is an Artinian ring, there exists positive integer t such that

$\mathfrak{m}^t \neq (0)$ and $\mathfrak{m}^{t+1} = (0)$. Let $\text{v.dim}_{R/\mathfrak{m}}(\mathfrak{m}^i/\mathfrak{m}^{i+1}) = k_i$ where $1 \leq i \leq t$. Since $|\mathfrak{m}^i| = |\mathfrak{m}^i/\mathfrak{m}^{i+1}| |\mathfrak{m}^{i+1}/\mathfrak{m}^{i+2}| \dots |\mathfrak{m}^t|$ and $k_i + \dots + k_t \leq |\mathbb{I}(\mathfrak{m}^i)|$, $|\mathfrak{m}^i| = q^{k_i + \dots + k_t} \leq q^{|\mathbb{I}(\mathfrak{m}^i)|}$. Also, $|R| = |R/\mathfrak{m}| |\mathfrak{m}| = q^{k_1 + \dots + k_t + 1} \leq q^{|\mathbb{I}(R)|}$. \square

We recall that in [15, Theorem 2], it was shown that for every positive integer g there exist finitely many finite rings with $\gamma(\Gamma(R)) = g$. But the result is not true when we replace $\gamma(\Gamma(R))$ with $\gamma(\mathbb{A}\mathbb{G}(R))$. In fact, for every finite field \mathbb{F} , the ring $R = \mathbb{F} \times \mathbb{F} \times \mathbb{F} \times \mathbb{F}$ has a toroidal annihilating-ideal graph (since $\gamma(\mathbb{A}\mathbb{G}(R)) = \gamma(\mathbb{A}\mathbb{G}(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2)) = 1$). Next, we proceed to show that for every integers $q > 0$ and $g \geq 0$, there are finitely many Artinian rings R , such that $\gamma(\mathbb{A}\mathbb{G}(R)) = g$ and $|R/\mathfrak{m}| \leq q$ for every maximal ideal \mathfrak{m} of R . We first prove a reduced form of the result.

Theorem 2.10 *For a prime power q and an integer $g \geq 0$, there are finitely many Artinian local rings (R, \mathfrak{m}) satisfying the following conditions:*

- (1) $\gamma(\mathbb{A}\mathbb{G}(R)) = g$,
- (2) $|R/\mathfrak{m}| = q$.

Proof. Suppose that R is an Artinian local ring with $|R/\mathfrak{m}| = q$ and $\gamma(\mathbb{A}\mathbb{G}(R)) = g$. Since there exists only one field of order q , we can assume that R is not a field. Thus there exists a positive integer t such that $\mathfrak{m}^t \neq (0)$, $\mathfrak{m}^{t+1} = (0)$. Since R is an Artinian ring and $|R/\mathfrak{m}| = q$, R is a finite ring. It is sufficient to show that $|R|$ is bounded by a constant depending only on g and q . The proof now proceeds by cases:

Case 1: $\text{v.dim}_{R/\mathfrak{m}}(\text{Ann}(\mathfrak{m})) \geq 2$. Since $\text{Ann}(\mathfrak{m})\mathfrak{m} = (0)$, $K_{|\mathbb{I}(R)|-5,3}$ is a subgraph of $\mathbb{A}\mathbb{G}(R)$. By the formula for the genus of complete bipartite graphs, $\lceil (|\mathbb{I}(R)| - 7)/4 \rceil \leq g$ thus $|\mathbb{I}(R)| \leq 4g + 7$ and therefore by Lemma 2.9, $|R| \leq q^{4g+7}$.

Case 2: $\text{v.dim}_{R/\mathfrak{m}}(\text{Ann}(\mathfrak{m})) = 1$.

Subcase 1: $t = 1$, i.e., $\mathfrak{m}^2 = (0)$. Since $|R| = |R/\mathfrak{m}| |\mathfrak{m}|$ and $\text{v.dim}_{R/\mathfrak{m}} \mathfrak{m} = 1$, we conclude that $|R| = q^2$.

Subcase 2: $t = 2$, i.e., $\mathfrak{m}^2 \neq (0)$ and $\mathfrak{m}^3 = (0)$. Let $\text{v.dim}_{R/\mathfrak{m}}(\mathfrak{m}/\mathfrak{m}^2) = k$. Then by Lemma 2.8, $\mathbb{A}\mathbb{G}(R)$ contains a copy of $K_{k-6,3}$. By the formula for the genus of complete bipartite graphs, $\lceil (k - 8)/4 \rceil \leq g$ thus $|k| \leq 4g + 8$. Since $|R| = |R/\mathfrak{m}| |\mathfrak{m}/\mathfrak{m}^2| |\mathfrak{m}^2|$ and $\text{v.dim}_{R/\mathfrak{m}} \mathfrak{m}^2 = 1$, we conclude that $|R| \leq q^{2+4g+8}$.

Subcase 3: $t \geq 3$. The proof will be divided into three steps.

Step 1: We prove that $|\mathfrak{m}^3|$ is bounded by a constant depending only on g and q . If $t = 3, 4, 5$, then $(\mathfrak{m}^3)^2 = (0)$. Therefore, $K_{|\mathbb{I}(\mathfrak{m}^3)|-1}$ is a subgraph of $\mathbb{A}\mathbb{G}(R)$ and by the formula for the genus of complete graphs, $\lceil (|\mathbb{I}(\mathfrak{m}^3)| - 5)/12 \rceil \leq g$ thus $|\mathbb{I}(\mathfrak{m}^3)| \leq 12g + 5$ and by Lemma 2.9, $|\mathfrak{m}^3| \leq q^{12g+5}$.

If $t \geq 5$ then $\mathfrak{m}^t \mathfrak{m}^3 = (0)$, $\mathfrak{m}^{t-1} \mathfrak{m}^3 = (0)$, $\mathfrak{m}^{t-2} \mathfrak{m}^3 = (0)$. Therefore, $K_{|\mathbb{I}(\mathfrak{m}^3)|-4,3}$ is a subgraph of $\mathbb{A}\mathbb{G}(R)$ and by the formula for the genus of complete bipartite graphs,

$\lceil (|I(\mathfrak{m}^3)| - 5)/4 \rceil \leq g$. Thus $|\mathbb{I}(\mathfrak{m}^3)| \leq 4g + 5$ and by Lemma 2.9, $|\mathfrak{m}^3| \leq q^{4g+5}$.

Step 2: We prove that $|\mathfrak{m}^2|$ is bounded by a constant depending only on g and q . If $\text{v.dim}_{R/\mathfrak{m}}(\mathfrak{m}^2/\mathfrak{m}^3) = 1$, then $|\mathfrak{m}^2| = q|\mathfrak{m}^3|$. Now, by Step 1, we conclude that $|\mathfrak{m}^2|$ is also bounded by a constant depending only on g and q . Suppose that $\text{v.dim}_{R/\mathfrak{m}}(\mathfrak{m}^2/\mathfrak{m}^3) = k \geq 2$. Let $\{x_1 + \mathfrak{m}^3, \dots, x_k + \mathfrak{m}^3\}$ be a basis for $\mathfrak{m}^2/\mathfrak{m}^3$ over R/\mathfrak{m} . Since $\mathfrak{m}^t \subsetneq Rx_i$, $|\mathbb{I}(Rx_i)| \geq 3$, for each i ($1 \leq i \leq k$). If $|\mathbb{I}(Rx_1)| = |\mathbb{I}(Rx_2)| = 3$, then since $Rx_i \cong R/\text{Ann}(x_i)$, $1 \leq i \leq 2$, there is no ideals between $\text{Ann}(x_i)$, $1 \leq i \leq 2$ and \mathfrak{m} . Now, if $\mathfrak{m}^2 \not\subseteq \text{Ann}(x_i)$, $1 \leq i \leq 2$, then $\text{Ann}(x_i) + \mathfrak{m}^2 = \mathfrak{m}$, $1 \leq i \leq 2$ and by Nakayama's lemma, $\text{Ann}(x_i) = \mathfrak{m}$, $1 \leq i \leq 2$, a contradiction. Therefore, $\mathfrak{m}^2 \subseteq \text{Ann}(x_i)$, $1 \leq i \leq 2$. Since $(\mathfrak{m}^2)(Rx_i) = (0)$, $1 \leq i \leq 2$ and $\mathfrak{m}^t \mathfrak{m}^2 = (0)$, $K_{|\mathbb{I}(\mathfrak{m}^2)|-2,3}$ is a subgraph of $\mathbb{A}\mathbb{G}(R)$. By the formula for genus of complete bipartite graphs, $\lceil (|\mathbb{I}(\mathfrak{m}^2)| - 4)/4 \rceil \leq g$, thus $|\mathbb{I}(\mathfrak{m}^2)| \leq 4g + 4$ and by Lemma 2.9, $|\mathfrak{m}^2| \leq q^{4g+4}$. Without loss of generality we can assume that $|\mathbb{I}(Rx_1)| \geq 4$. Then $K_{|\mathbb{I}(\text{Ann}(x_1))|-4,3}$ is a subgraph of $\mathbb{A}\mathbb{G}(R)$. By the formula for genus of complete bipartite graphs $\lceil (|\mathbb{I}(\text{Ann}(x_1))| - 6)/4 \rceil \leq g$, thus $|\mathbb{I}(\text{Ann}(x_1))| \leq 4g + 6$. Since $|\mathbb{I}(\text{Ann}(x_1) \cap \mathfrak{m}^2)| \leq |\mathbb{I}(\text{Ann}(x_1))|$ and dimension of every vector space less than or equal to cardinality of each its generating set, we conclude that $\text{v.dim}_{R/\mathfrak{m}}((\text{Ann}(x_1) \cap \mathfrak{m}^2) + \mathfrak{m}^3)/\mathfrak{m}^3 \leq |\mathbb{I}(\text{Ann}(x_1))|$. Let $\theta : \mathfrak{m}^2 \rightarrow \mathfrak{m}^2 x_1$ be the group homomorphism defined by multiplication by x_1 . Then we have $|\mathfrak{m}^2|/|\mathfrak{m}^2 \cap \text{Ann}(x_1)| = |\mathfrak{m}^2 x_1|$. It follows that $|\mathfrak{m}^2| \leq |\mathfrak{m}^3| |\mathfrak{m}^2 \cap \text{Ann}(x_1)| \leq |\mathfrak{m}^3| |\mathfrak{m}^2 \cap \text{Ann}(x_1) + \mathfrak{m}^3| \leq |\mathfrak{m}^3|^2 q^{|\mathbb{I}(\text{Ann}(x_1))|} \leq |\mathfrak{m}^3|^2 q^{4g+6}$. Now, by Step 1, we conclude that $|\mathfrak{m}^2|$ is also bounded by a constant depending only on g and q .

Step 3: Now, we show that $|\mathfrak{m}|$ is bounded by a constant depending only on g and q . We can assume that $\text{v.dim}_{R/\mathfrak{m}}(\mathfrak{m}/\mathfrak{m}^2) = k \geq 5$. Let $\{x_1 + \mathfrak{m}^2, \dots, x_k + \mathfrak{m}^2\}$ be a basis for $\mathfrak{m}/\mathfrak{m}^2$ over R/\mathfrak{m} . Since $\mathfrak{m}^t \subsetneq Rx_i$, $|\mathbb{I}(Rx_i)| \geq 3$, for each i ($1 \leq i \leq k$). If $|\mathbb{I}(Rx_1)| = |\mathbb{I}(Rx_2)| = 3$, then since $Rx_i \cong R/\text{Ann}(x_i)$, $1 \leq i \leq 2$, there is no ideals between $\text{Ann}(x_i)$, $1 \leq i \leq 2$ and \mathfrak{m} . If $\mathfrak{m}^2 \not\subseteq \text{Ann}(x_i)$, for $i = 1$ or 2 , then $\text{Ann}(x_i) + \mathfrak{m}^2 = \mathfrak{m}$, and so by Nakayama's lemma, $\text{Ann}(x_i) = \mathfrak{m}$. Since $\text{v.dim}_{R/\mathfrak{m}}(\text{Ann}(\mathfrak{m})) = 1$ and $\mathfrak{m}^t \mathfrak{m} = (0)$, $\text{Ann}(\mathfrak{m}) = \mathfrak{m}^t$. Thus $Rx_i \subseteq \mathfrak{m}^t \subseteq \mathfrak{m}^2$, a contradiction. Therefore, $\mathfrak{m}^2 \subseteq \text{Ann}(x_1) \cap \text{Ann}(x_2)$. If $\text{v.dim}_{R/\mathfrak{m}}(\text{Ann}(x_i)/\mathfrak{m}^2) \leq k - 2$ for $i = 1$ or 2 , then there exists proper ideal $I \subseteq R$ such that $\text{Ann}(x_i)/\mathfrak{m}^2 \subsetneq I/\mathfrak{m}^2 \subsetneq \mathfrak{m}/\mathfrak{m}^2$, and hence $\text{Ann}(x_i) \subsetneq I \subsetneq \mathfrak{m}$, a contradiction. Thus $\text{v.dim}_{R/\mathfrak{m}}(\text{Ann}(x_i)/\mathfrak{m}^2) \geq k - 1$ for $i = 1, 2$. It follows that $\text{v.dim}_{R/\mathfrak{m}}(\text{Ann}(x_1) \cap \text{Ann}(x_2)) \geq k - 2$. Since $Rx_1(\text{Ann}(x_1) \cap \text{Ann}(x_2)) = (0)$, $Rx_2(\text{Ann}(x_1) \cap \text{Ann}(x_2)) = (0)$ and $\mathfrak{m}^t(\text{Ann}(x_1) \cap \text{Ann}(x_2)) = (0)$ and $|\mathbb{I}(\text{Ann}(x_1) \cap \text{Ann}(x_2)) \setminus \{(0), Rx_1, Rx_2\}| \geq k - 5$, we conclude that $K_{k-5,3}$ is a subgraph of $\mathbb{A}\mathbb{G}(R)$. Now, by the formula for genus of complete bipartite graphs, $k \leq 5g + 6$ hence $|\mathfrak{m}| \leq |\mathfrak{m}^2|^2 q^{5g+6}$. Thus by Step 2, we conclude that $|\mathfrak{m}|$ is also bounded by a constant depending only on g and q .

Finally, without loss of generality we can assume that $|\mathbb{I}(Rx_1)| \geq 4$. Thus $K_{|\mathbb{I}(\text{Ann}(x_1))|-4,3}$

is a subgraph of $\mathbb{A}\mathbb{G}(R)$. By the formula for genus of complete bipartite graphs, $\lceil (|\mathbb{I}(\text{Ann}(x_1))| - 6)/4 \rceil \leq g$ and hence $|\mathbb{I}(\text{Ann}(x_1))| \leq 4g + 6$. Since $|\mathbb{I}(\text{Ann}(x_1) \cap \mathfrak{m}^2)| \leq |\mathbb{I}(\text{Ann}(x_1))|$ and the dimension of every vector space less than or equal to cardinality of each its generating set, we conclude that $\text{v.dim}_{R/\mathfrak{m}}(\text{Ann}(x_1) + \mathfrak{m}^2)/\mathfrak{m}^2 \leq |\mathbb{I}(\text{Ann}(x_1))|$. Let $\vartheta : \mathfrak{m} \rightarrow \mathfrak{m}x_1$ be the group homomorphism defined by multiplication by x_1 . Then $|\mathfrak{m}| = |\mathfrak{m}x_1| |\text{Ann}(x_1)| \leq |\mathfrak{m}^2| |\text{Ann}(x_1) + \mathfrak{m}^2| \leq |\mathfrak{m}^2|^2 q^{|\mathbb{I}(\text{Ann}(x_1))|} \leq |\mathfrak{m}^2|^2 q^{4g+6}$. Now, by Step 2, we conclude that $|\mathfrak{m}|$ is also bounded by a constant depending only on g and q . This finishes the proof, since $|R| = |R/\mathfrak{m}| |\mathfrak{m}|$ and so $|R|$ is bounded by a constant depending only on g and q . \square

Corollary 2.11 *For integers $q > 0$ and $g \geq 0$, there are finitely many Artinian rings R satisfying the following conditions:*

- (1) $\gamma(\mathbb{A}\mathbb{G}(R)) = g$,
- (2) $|R/\mathfrak{m}| \leq q$ for any maximal ideal \mathfrak{m} of R .

Proof. Let R be an Artinian ring with $\gamma(\mathbb{A}\mathbb{G}(R)) = g$. It is well known that $R \cong R_1 \times \dots \times R_n$ for some positive integer n , where every R_i ($i = 1, 2, \dots, n$), is an Artinian local ring. Since $|R_i/\mathfrak{m}_i| \leq q$, every R_i is a finite local ring. For ease of exposition, we assume that $|R_i| \geq |R_{i+1}|$ for $i = 1, 2, \dots, n-1$ as there is no loss of generality with this assumption.

It sufficient to show that $|R|$ is bounded by a constant depending only on g and q . Furthermore, $|R_1| \geq |R_i|$ for all i , it is enough to bound R_1 by a constant depending only on g and q . By a slight abuse of notation, we denote the set $0 \times \dots \times R_i \times \dots \times 0$ by R_i , and the set $R_1 \times \dots \times R_{i-1} \times 0 \times R_{i+1} \times \dots \times R_n$ by \bar{R}_i .

If $n \geq 3$, then $|\mathbb{I}(\bar{R}_1^*)| \geq 3$. Thus $\mathbb{A}\mathbb{G}(R)$ contains a copy of $K_{|\mathbb{I}(\bar{R}_1)|-1,3}$. By the formula for the genus of complete bipartite graphs, we have $\lceil (|\mathbb{I}(\bar{R}_1)| - 3)/4 \rceil \leq g$ so $|\mathbb{I}(\bar{R}_1)| \leq 4g + 3$ and thus by Lemma 2.9, $|R_1| \leq q^{4g+3}$.

Suppose that $n = 2$. Let $|\mathbb{I}(R_2)| \geq 4$, this means $\mathbb{A}\mathbb{G}(R)$ contains a copy of $K_{|\mathbb{I}(R_1)|-1,3}$, which as above gives $|R_1| \leq q^{4g+3}$.

Let $|\mathbb{I}(R_2)| \leq 3$. If R_1 be a field, then $|R_1| \leq q$. Thus we may assume that R_1 is not a field. Let $\mathfrak{m} \neq (0)$ be the maximal ideal of R_1 . Since R_1 is an Artinian ring, $\text{Ann}(\mathfrak{m}) \neq (0)$. Assume that $I = (\mathfrak{m}, 0)$, $J_1 = (\text{Ann}(\mathfrak{m}), 0)$, $J_2 = (\text{Ann}(\mathfrak{m}), R_2)$ and $J_3 = (0, R_2)$. Then $IJ_i = (0)$ for $1 \leq i \leq 3$. Therefore, $\mathbb{A}\mathbb{G}(R)$ contains a copy of $K_{|\mathbb{I}(R_1)|-2,3}$. Now, by the formula for the genus of complete bipartite graphs, $\lceil (|\mathbb{I}(R_1)| - 4)/4 \rceil \leq g$ and so $|\mathbb{I}(R_1)| \leq 4g + 4$. Therefore, by Lemma 2.9, $|R_1| \leq q^{4g+4}$. Finally, suppose that $n = 1$. Let R be an Artinian local ring with $|R/\mathfrak{m}| = q$. Then by Theorem 2.10, $|R|$ is bounded by a constant depending only on g and q . \square

3 The Genus of Annihilating-Ideal Graphs of Noetherian Rings

In this section, we investigate the genus of annihilating-ideal graphs of Noetherian rings.

Let R be a ring. We say that the annihilating-ideal graph $\mathbb{AG}(R)$ has assenting chain condition (ACC) (resp., descending chain condition (DCC)) on vertices if R has ACC (resp., DCC) on $\mathbb{A}(R)$.

First, we need the following useful lemmas.

Lemma 3.1 [6, Theorem 1.1] *Let R be a ring that is not a domain. Then $\mathbb{AG}(R)$ has ACC (resp., DCC) on vertices if and only if R is a Noetherian (resp., an Artinian) ring.*

Lemma 3.2 [6, Proposition 1.7] *Let R be a Noetherian ring. If all non-trivial ideals of R are vertices of $\mathbb{AG}(R)$, then R has finitely many maximal ideals.*

By Lemma 2.1, any non-trivial ideal of an Artinian ring R is a vertex of $\mathbb{AG}(R)$. Now, a natural question is posed: If R is a Noetherian ring and all non-trivial ideals of R are vertices of $\mathbb{AG}(R)$, then is R an Artinian ring? the answer of this question is negative (see [1, Example 17]). Next, we have the following result.

Proposition 3.3 *Let (R, \mathfrak{m}) be a Gorenstein ring. Then all non-trivial ideals of R are vertices of $\mathbb{AG}(R)$.*

Proof. If (R, \mathfrak{m}) is a Gorenstein ring, then $\text{v.dim}_{R/\mathfrak{m}}(\text{Ann}(\mathfrak{m})) = 1$. It follows that $\text{Ann}(\mathfrak{m}) \neq (0)$ and so \mathfrak{m} is a vertex of $\mathbb{AG}(R)$, i.e., all non-trivial ideals of R are vertices of $\mathbb{AG}(R)$. \square

Proposition 3.4 *Let R be a Noetherian local ring. If $\gamma(\mathbb{AG}(R)) < \infty$, then either R is a domain or all non-trivial ideals of R are vertices of $\mathbb{AG}(R)$.*

Proof. Let R be a Noetherian ring and $\gamma(\mathbb{AG}(R)) < \infty$. We can assume that R is not a domain. If R is an Artinian ring, then by Lemma 2.1, every ideal of R is a vertex of $\mathbb{AG}(R)$. Suppose that R is not an Artinian ring. By Lemma 3.1, there exists an infinite descending chain $I_1 \supsetneq I_2 \supsetneq \dots \supsetneq I_n \supsetneq \dots$ in $\mathbb{A}(R)$. Suppose that P is a maximal element in $\mathbb{A}(R)$ such that $I_1 \subseteq P$. Clearly P is maximal among all annihilators of non-zero elements of R and so by [9, Theorem 6], P is a prime ideal. Also, $\{I_n \mid n \in \mathbb{N}\} \subseteq P$ and

hence $|\mathbb{I}(P)^*| = \infty$. Set $I = \text{Ann}(P)$. If $|\mathbb{I}(I)^*| \geq 3$, then $K_{|\mathbb{I}(I)^*|, |\mathbb{I}(P)^*|}$ is a subgraph of $\mathbb{A}\mathbb{G}(R)$. Since $|\mathbb{I}(P)^*| = \infty$, $\gamma(\mathbb{A}\mathbb{G}(R)) = \infty$, a contradiction. If $|\mathbb{I}(I)^*| = 2$, then we must have $I = Rx$ for some $x \in R$, for otherwise, there are two nonzero distinct ideals $Rz, Rx \subseteq I$, and so $|\mathbb{I}(I)^*| \geq 3$, a contradiction. Since $\text{Ann}(I) = \text{Ann}(x) = P$, $Rx \cong R/\text{Ann}(x)$ and Rx has only one nonzero R -submodule, we conclude that there is no ideals between P and \mathfrak{m} . Since $P \subseteq P + \mathfrak{m}^2 \subseteq \mathfrak{m}$, either $P + \mathfrak{m}^2 = P$ or $P + \mathfrak{m}^2 = \mathfrak{m}$. If $P + \mathfrak{m}^2 = P$, then $\mathfrak{m}^2 \subseteq P$. It follows that $P = \mathfrak{m}$. Also, if $P + \mathfrak{m}^2 = \mathfrak{m}$, then by Nakayama's lemma, $P = \mathfrak{m}$. Thus $I = Rx$ is a simple R -module, contrary to $|\mathbb{I}(I)^*| = 2$. Thus $|\mathbb{I}(I)^*| = 1$, i.e., I is a simple R -module. This implies that $P = \mathfrak{m}$. Thus \mathfrak{m} is a vertex of $\mathbb{A}\mathbb{G}(R)$, i.e., all non-trivial ideals of R are vertices of $\mathbb{A}\mathbb{G}(R)$. \square

Theorem 3.5 *Let R be a Noetherian ring such that all non-trivial ideals of R are vertices of $\mathbb{A}\mathbb{G}(R)$. If $\gamma(\mathbb{A}\mathbb{G}(R)) < \infty$, then either R is a Gorenstein ring or R is an Artinian ring with finitely many ideals.*

Proof. Suppose that R is a Noetherian ring, all non-trivial ideals of R are vertices of $\mathbb{A}\mathbb{G}(R)$ and $\gamma(\mathbb{A}\mathbb{G}(R)) < \infty$. Suppose that R is a local ring with maximal ideal \mathfrak{m} . We can assume that R is not a Gorenstein ring. Therefore $\text{v.dim}_{R/\mathfrak{m}}(\text{Ann}(\mathfrak{m})) \geq 2$. It follows that $|\text{Ann}(\mathfrak{m})^*| \geq 3$ and $\text{Ann}(\mathfrak{m})\mathfrak{m} = (0)$. If $|\mathbb{I}(R)| = \infty$, then we conclude that $K_{|\mathbb{I}(R)|-5,3}$ is a subgraph of $\mathbb{A}\mathbb{G}(R)$. Thus by the formula for the genus of complete bipartite graphs, we have $|\mathbb{I}(R)| = \infty$, a contradiction. Thus $|\mathbb{I}(R)| < \infty$, R , i.e., R is an Artinian ring with finitely many ideals.

Now, we can assume that R is not a local ring. It is sufficient to shown that every prime ideal is a maximal ideal. By Lemma 3.2, R has finitely many maximal ideals. Suppose that $k \geq 2$, $\mathfrak{m}_1, \dots, \mathfrak{m}_k$ are all maximal ideals of R and $J(R) = \mathfrak{m}_1 \cap \dots \cap \mathfrak{m}_k$. Consider the following chain

$$J(R) \supseteq J(R)^2 \supseteq \dots \supseteq J(R)^n \supseteq \dots$$

Suppose that $J(R)^i \supsetneq J(R)^{i+1}$ for each $i \geq 1$. Then $|\mathbb{I}(J(R))| = \infty$. Since every ideal of R is a vertex of $\mathbb{A}\mathbb{G}(R)$, we conclude that $I_1 = \text{Ann}(\mathfrak{m}_1) \neq (0)$ and $I_2 = \text{Ann}(\mathfrak{m}_2) \neq (0)$. Also, $I_1 \not\subseteq I_2$ and $I_2 \not\subseteq I_1$ (otherwise, either $(0) = I_1(\mathfrak{m}_1 + \mathfrak{m}_2) = I_1R$ or $(0) = I_2(\mathfrak{m}_1 + \mathfrak{m}_2) = I_2R$, a contradiction). Clearly $I_1J(R) = I_2J(R) = (I_1 + I_2)J(R) = (0)$ and so $K_{3, |\mathbb{I}(J(R))^*|}$ is a subgraph of $\mathbb{A}\mathbb{G}(R)$. Since $|\mathbb{I}(J(R))| = \infty$, $\gamma(\mathbb{A}\mathbb{G}(R)) = \infty$, a contradiction. Therefore $J(R)^n = J(R)^{n+1}$ for some $n \geq 1$. Thus by Nakayama's lemma, $J(R)^n = (0)$. Suppose that P is a prime ideal of R , since $J(R)^n = (0) \subseteq P$, there exists \mathfrak{m}_i , $1 \leq i \leq n$ such that $\mathfrak{m}_i = P$. Thus every prime ideal of R is a maximal ideal of R . Thus R is an Artinian ring and by Theorem 2.7, R has finitely many ideals. \square

We conclude this paper with the following corollary.

Corollary 3.6 *Let R be a non-domain Noetherian local ring. If $\gamma(\mathbb{A}\mathbb{G}(R)) < \infty$, then either R is a Gorenstein ring or R is an Artinian ring with finitely many ideals.*

Proof. By Proposition 3.4 and Theorem 3.5 is clear. \square

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